

SYMMETRIC FUNCTIONS WITH NON-TRIVIAL ARITY GAP

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ABSTRACT. Given an n -ary k -valued function f , $\text{gap}(f)$ denotes the essential arity gap of f which is the minimal number of essential variables in f which become fictive when identifying any two distinct essential variables in f . The class $G_{p,k}^n$ of all n -ary k -valued functions f with $2 \leq \text{gap}(f)$ is explicitly determined by the authors and R. Willard in [5, 7, 8].

We prove that if $f \in G_{p,k}^n$ is a symmetric function with non-trivial arity gap and $g = f(x_i = c_i)$ is a subfunction of f , where c_i is a constant, then $g \in G_{n-1,k}^{n-1} \cup G_{2,k}^{n-1}$ or g is a constant function. A deep description of the class of all symmetric linear k -valued functions with non-trivial arity gap is given.

INTRODUCTION

Given a function f , the essential variables in f are defined as variables which occur in f and weigh with the values of that function.

In this paper we obtain some results concerning simplifying of functions by identification of essential variables.

The essential arity gap (gap) of Boolean functions are deeply investigated in [5]. In [8], R. Willard proved that if $n > k$ then $\text{gap}(f) \leq 2$.

1. PRELIMINARIES

Let k be a natural number with $k \geq 2$. Denote by $K = \{0, 1, \dots, k-1\}$ the set (ring) of remainders modulo k . An n -ary k -valued function (operation) on K is a mapping $f : K^n \rightarrow K$ for some natural number n , called the *arity* of f . The set of all such functions is denoted by P_k^n .

Definition 1. Let $X_n = \{x_1, \dots, x_n\}$ be the set of n variables. A variable x_i is called the *essential variable* in f , or f *essentially depends* on x_i , if there exist values $a_1, \dots, a_n, b \in K$, such that

$$f(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \neq f(a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n).$$

The set of all essential variables in a function f is denoted by $\text{Ess}(f)$ and the number of its essential variables is denoted by $\text{ess}(f) = |\text{Ess}(f)|$.

Definition 2. Let $x_{i_1}, \dots, x_{i_m} \in \text{Ess}(f)$ be m , $m \leq n$, essential variables in f and c_{i_1}, \dots, c_{i_m} be m , constants from K . The function $g = f(x_{i_1} = c_{i_1}, \dots, x_{i_m} = c_{i_m})$ obtained from f by replacing the variables x_{i_1}, \dots, x_{i_m} with constants c_{i_1}, \dots, c_{i_m} is called a *subfunction* of f with respect to the variables $M = \{x_{i_1}, \dots, x_{i_m}\}$ and constants $C = \{c_{i_1}, \dots, c_{i_m}\}$, briefly a *subfunction* of f .

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We shall say that g is a subfunction of f of order m . When g is a subfunction of f , we shall write $g \prec f$. Note that g is an $(n - m)$ -ary k -valued function, where $m = |M|$.

We shall use the following denotations: $Sub(f) := \{g \mid g \prec f\}$ and $sub(f) := |Sub(f)|$.

Definition 3. A set M of essential variables in f is called separable in f if there is a subfunction g of f such that $M = Ess(g)$.

Let x_i and x_j be two distinct essential variables in f . The function h is obtained from $f \in P_k^n$ by the identification of the variable x_i with x_j , if

$$h(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) = f(a_1, \dots, a_{i-1}, a_j, a_{i+1}, \dots, a_n),$$

for all $(a_1, \dots, a_n) \in K^n$.

Briefly, when h is obtained from f , by identification of the variable x_i with x_j , we will write $h = f_{i \leftarrow j}$ and h is called the identification minor of f . $Min(f)$ denotes the set of all identification minors of f .

We shall allow formation of identification minors when x_i or x_j are not essential in f . Such minors of f are *trivial* and they do not belong to $Min(f)$. So, if x_i is not essential in f , then we define $f_{i \leftarrow j} := f$.

Clearly, $ess(f_{i \leftarrow j}) \leq ess(f)$, because $x_i \notin Ess(f_{i \leftarrow j})$, even though it might be essential in f .

Definition 4. Let $f \in P_k^n$ be an n -ary k -valued function. Then the essential arity gap (shortly arity gap or gap) of f is defined by

$$gap(f) := ess(f) - \max_{h \in Min(f)} ess(h).$$

We let $G_{p,k}^m$ denote the set of all functions in P_k^n which essentially depend on m variables whose arity gap is p , i.e. $G_{p,k}^m = \{f \in P_k^n \mid ess(f) = m \text{ \& } gap(f) = p\}$, with $m \leq n$.

Several problems concerning $gap(f)$ for Boolean functions are discussed in the work of K. Chimev, O. Lupanov and A. Salomaa [1, 3, 4]. It is proved that $gap(f) \leq 2$, when $f \in P_2^n$, $n \geq 2$.

This result is generalized for arbitrary finite valued functions as follows: $gap(f) \leq k$ for all $f \in P_k^n$.

In [5], the Boolean functions whose arity gap is non-trivial are studied and the class $G_{2,2}^n$ is completely described.

The case $2 \leq p \leq n$ and $n > k$ is fully described in [8] where it is proved that if $f \in G_{2,k}^n$ then $gap(f) \leq 2$, f is a symmetric function and the identification minors $f_{i \leftarrow j}$ do not depend on x_j for all j, i with $1 \leq j < i \leq n$.

In [6] and [7], we have explicitly described the set of functions f whose arity gap is m with $m \geq 2$, $n < k$.

Given a variable x and $\alpha \in K$, x^α is a function defined by:

$$x^\alpha = \begin{cases} 1 & \text{if } x = \alpha \\ 0 & \text{if } x \neq \alpha. \end{cases}$$

We shall use *sums of conjunctions* (*SC*) for representation of functions in P_k^n . This is the most natural representation of the functions in finite algebras. It is based on so called operation tables of the functions.

Each function $f \in P_k^n$ can be uniquely represented in SC-form as follows

$$f = a_0.x_1^0 \dots x_n^0 \oplus \dots \oplus a_m.x_1^{\alpha_1} \dots x_n^{\alpha_n} \oplus \dots \oplus a_{k^n-1}.x_1^{k-1} \dots x_n^{k-1}$$

with $m = \sum_{i=1}^n \alpha_i k^{n-i}$, and $\alpha_i, a_m \in K$, where " \oplus " and "." are the operations addition and multiplication modulo k in the ring K .

Let g be a subfunction of f and $\text{gap}(f) \geq 2$. We say that g inherit the gap of f if $\text{gap}(g) \geq 2$.

For a Boolean variable x we denote by $\neg(x)$ the unary operation *negation*, i.e.

$$\neg x = x^0 = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1. \end{cases}$$

Example 1. First, we consider the class of binary and ternary Boolean functions.

The most simple case is when a function belongs to the class $G_{2,2}^2$. Then all its subfunctions depend essentially on at most one variable. This case can be considered as trivial, one.

Next, we shall pay attention to the case $f \in G_{2,2}^3$. From Theorem 3.2 [5] the function f has to be represented in one of the following special forms

$$(1) \quad f = x_3^\alpha (x_1^0 x_2^1 \oplus x_1^1 x_2^0) \oplus x_1^\beta x_2^\beta,$$

or

$$(2) \quad f = x_3^\alpha (x_1^0 x_2^0 \oplus x_1^1 x_2^1) \oplus x_3^{\neg(\alpha)} (x_1^0 x_2^1 \oplus x_1^1 x_2^0),$$

where $\alpha, \beta \in \{0, 1\}$.

Let $f \in G_{2,2}^3$ be a function represented as (1). It is easy to see that f is not a symmetric function. Then we have the following two subfunctions of f with respect to x_3 :

$$f(x_3 = \alpha) = x_1^0 x_2^1 \oplus x_1^1 x_2^0 \oplus x_1^\beta x_2^\beta \notin G_{2,2}^2 \quad \text{and} \quad f(x_3 = \neg\alpha) = x_1^\beta x_2^\beta \notin G_{2,2}^2.$$

Hence these subfunctions do not inherit the arity gap of f .

The subfunctions of f with respect to x_1 are :

$$f(x_1 = 0) = x_3^\alpha x_2^1 \oplus 0^\beta x_2^\beta \quad \text{and} \quad f(x_1 = 1) = x_3^\alpha x_2^0 \oplus 1^\beta x_2^\beta.$$

Without any difficulties, it might be seen that these subfunctions belong to $G_{2,2}^2$ if and only if $\alpha \neq \beta$. By symmetry we obtain the same result for the subfunctions of f with respect to x_2 . Summarizing this result we conclude that the Boolean functions represented by (1) have exactly four subfunctions which inherit the gap of f when $\alpha \neq \beta$.

Next, let $f \in G_{2,2}^3$ be a function represented as in (2). It is not so difficult to see that f is a symmetric function, and it is enough to consider the subfunctions of f with respect to one variable, only. Then we have the following two subfunctions of f with respect to x_1 :

$$(3) \quad f(x_1 = \alpha) = x_2^0 x_3^0 \oplus x_2^1 x_3^1 \quad \text{and} \quad f(x_1 = \neg\alpha) = x_2^0 x_3^1 \oplus x_2^1 x_3^0,$$

which belong to $G_{2,2}^3$. By symmetry it follows that all subfunctions of f of order 1 have non-trivial arity gap and they inherit this property of f . Hence each function represented in form (2) has exactly six subfunctions of order 1 which inherit the arity gap of f .

2. SYMMETRIC FUNCTIONS WITH NON-TRIVIAL ARITY GAP

We shall study the behavior of the symmetric k -valued functions f with non-trivial arity gap, i.e. with $\text{gap}(f) > 1$. As usual we shall denote by S_n the set of all permutations of the set $\{1, \dots, n\}$.

Definition 5. Let $f \in P_k^n$ be a function such that

$$\text{Ess}(f) = \{x_{i_1}, \dots, x_{i_m}\} \subseteq \{x_1, \dots, x_n\}.$$

Let S_f be the set of all permutations of $\{i_1, \dots, i_m\}$. We say that f is a symmetric function with respect to its essential variables if

$$f(x_1, \dots, x_n) = f(x_{\psi_\pi(1)}, \dots, x_{\psi_\pi(n)})$$

where $\psi_\pi \in S_n$ is defined as follows

$$\psi_\pi(j) = \begin{cases} \pi(j) & \text{if } j \in \{i_1, \dots, i_m\} \\ j & \text{otherwise,} \end{cases}$$

for all $\pi \in S_f$.

Lemma 1. Let $2 \leq p \leq k$ and $2 \leq p \leq n$. If $f \in G_{p,k}^n$ is a symmetric function, then $p = 2$ or $p = n$.

Proof. Indeed, suppose this is were not the case. Then $2 < p < n$. Hence there exists an identification minor h of f such that $\text{gap}(f) = n - \text{ess}(h)$ and $2 < n - \text{ess}(h) < n$. Without loss of generality assume that $h = f_{n \leftarrow n-1}$ and $\text{Ess}(h) = \{x_1, \dots, x_q\}$, where $q = n - p$ with $0 < q < n - 2$. Then $x_{n-2} \in \text{Ess}(f) \setminus \text{Ess}(h)$. Hence for all $c_1, \dots, c_{n-3}, c_{n-2}, d_{n-2}, c_{n-1} \in K$ we have

$$h(c_1, \dots, c_{n-3}, c_{n-2}, c_{n-1}, c_{n-1}) = h(c_1, \dots, c_{n-3}, d_{n-2}, c_{n-1}, c_{n-1}),$$

i.e.

$$f(c_1, \dots, c_{n-3}, c_{n-2}, x_{n-1}, x_{n-1}) = f(c_1, \dots, c_{n-3}, d_{n-2}, x_{n-1}, x_{n-1}).$$

Let $\pi \in S_n$ be a permutation of the set $\{1, \dots, n\}$ defined as follows

$$\pi(i) = \begin{cases} 1 & \text{if } i = n - 2 \\ n - 2 & \text{if } i = 1 \\ i & \text{otherwise.} \end{cases}$$

Since f is symmetric we obtain

$$f(c_{n-2}, c_2, \dots, c_1, x_{n-1}, x_{n-1}) = f(d_{n-2}, c_2, \dots, c_1, x_{n-1}, x_{n-1})$$

for all $c_1, c_2, \dots, c_{n-2}, d_{n-2} \in K$. Hence $x_1 \notin \text{Ess}(h)$, which is a contradiction. \square

Lemma 2. Let $3 < n \leq k$. If $f \in G_{2,k}^n$ is a symmetric function then $x_v \notin \text{Ess}(f_{u \leftarrow v})$ for all $1 \leq u, v \leq n$ with $u \neq v$.

Proof. From Lemma 4.1 [7] there are $1 \leq u, v \leq n$ with $u \neq v$ such that $x_v \notin \text{Ess}(f_{u \leftarrow v})$. Suppose that there is a pair $i, j \in \{1, \dots, n\}$, $i \neq j$ such that $x_j \in \text{Ess}(f_{i \leftarrow j})$. Clearly, $(u, v) \neq (i, j)$.

There are two possible cases.

First, without loss of generality let us assume that $i = u = 1$, $j = 2$ and $v = 3$. Then there are constants $c_1, c_2, \dots, c_n \in K$ such that

$$f(c_1, c_1, c_3, \dots, c_n) \neq f(c_2, c_2, c_3, \dots, c_n)$$

and

$$f(c, x_2, c, x_4, \dots, x_n) = f(d, x_2, d, x_4, \dots, x_n)$$

for all $c, d \in K$. By the symmetry of f we obtain

$$f(c, c, x_2, x_4, \dots, x_n) = f(d, d, x_2, x_4, \dots, x_n)$$

for all $c, d \in K$. In particular, we have

$$f(c_1, c_1, x_2, x_4, \dots, x_n) = f(c_2, c_2, x_2, x_4, \dots, x_n),$$

which is a contradiction.

Second, again without loss of generality let us assume $u = 1$, $v = 2$, $i = 3$ and $j = 4$. Then we have

$$f(c, c, x_3, x_4, \dots, x_n) = f(d, d, x_3, x_4, \dots, x_n)$$

for all $c, d \in K$ and there exist constants $c_1, c_2, \dots, c_n \in K$ such that

$$f(c_1, c_2, c_3, c_3, c_5, \dots, c_n) \neq f(c_1, c_2, c_4, c_4, c_5, \dots, c_n).$$

By the symmetry of f we have

$$f(c_1, c_2, c_3, c_3, c_5, \dots, c_n) = f(c_3, c_3, c_1, c_2, c_5, \dots, c_n) =$$

$$f(c_4, c_4, c_1, c_2, c_5, \dots, c_n) = f(c_1, c_2, c_4, c_4, c_5, \dots, c_n),$$

which is a contradiction. \square

The next lemma might be proved as a direct corollary of Theorem 4.4 [7].

Lemma 3. *Let $3 < n \leq k$. If $f \in G_{2,k}^n$ and $x_v \notin \text{Ess}(f_{u \leftarrow v})$ for all $1 \leq u, v \leq n$ with $u \neq v$, then $f = t \oplus g$, where t is a symmetric function with $\text{Ess}(t_{u \leftarrow v}) = \text{Ess}(f) \setminus \{x_v, x_u\}$ for all $1 \leq u, v \leq n$, $u \neq v$ and $g \in G_{n,k}^n$ with $g_{i \leftarrow j} = 0$ for all $1 \leq i, j \leq n$, $i \neq j$ or $g = 0$.*

Let k and n , $k \geq n > 1$ be two natural numbers and $K^n = \{\hat{\alpha} | \hat{\alpha} = (\alpha_1, \dots, \alpha_n), \alpha_i \in K, i = 1, \dots, n\}$ be the set of all n -tuples (strings) over K .

For each n , $k \geq n > 1$, the set K^n is divided in two subsets as follows:

$$Eq_k^n := \{(\alpha_1, \dots, \alpha_n) \in K^n \mid \alpha_i = \alpha_j, \text{ for some } i, j \leq n, i \neq j\}$$

and

$$Dis_k^n := \{(\alpha_1, \dots, \alpha_n) \in K^n \mid \alpha_i \neq \alpha_j, \text{ for all } i, j \leq n, i \neq j\}.$$

Lemma 4. *If $f \in G_{p,k}^n$, $2 \leq n$ is a symmetric function with $p = n$ or n is an even natural number, then*

$$f(c_1, \dots, c_1) = f(c_2, \dots, c_2)$$

for all $c_1, c_2 \in K$.

Proof. Let us assume that $p = n$ and suppose that

$$f(c_1, \dots, c_1) \neq f(c_2, \dots, c_2)$$

for some $c_1, c_2 \in K$. Then we have $f_{2 \leftarrow 1}(c_1, \dots, c_1) \neq f_{2 \leftarrow 1}(c_2, \dots, c_2)$, i.e. $f_{2 \leftarrow 1}$ depends on at least one variable which is a contradiction because $p = n$.

Let $p = 2 < n$ and n is an even natural number. Let $c_1, c_2 \in K$ be arbitrary two constants and let us assume the non-trivial case $c_1 \neq c_2$. From Lemma 2 it follows that $x_v \notin \text{Ess}(f_{u \leftarrow v})$ for all $1 \leq u, v \leq n$ with $u \neq v$. Then we obtain

$$\begin{aligned}
& f(c_2, c_2, \dots, c_2) \\
&= f(c_1, c_1, c_2, c_2, c_2, \dots, c_2) && \text{because } x_2 \notin \text{Ess}(f_{1 \leftarrow 2}) \\
&= f(c_1, c_1, c_1, c_1, c_2, \dots, c_2) && \text{because } x_3 \notin \text{Ess}(f_{4 \leftarrow 3}) \\
&= f(c_1, c_1, c_1, c_1, c_1, c_1, c_2, \dots, c_2) && \text{because } x_5 \notin \text{Ess}(f_{6 \leftarrow 5}) \\
&\dots \quad \dots && \dots \quad \dots \\
&= f(c_1, c_1, \dots, c_1, c_1, c_2, c_2) && \text{because } x_{n-3} \notin \text{Ess}(f_{n-2 \leftarrow n-3}) \\
&= f(c_1, c_1, \dots, c_1, c_1, c_1, c_1) && \text{because } x_{n-1} \notin \text{Ess}(f_{n \leftarrow n-1}).
\end{aligned}$$

□

Remark 1. Note that Theorem 3.1 [7] implies that $f \in G_{n,k}^n$, if and only if f can be represented as follows

$$(4) \quad f = \left[\bigoplus_{\hat{\beta} \in \text{Dis}_k^n} a_{\hat{\beta}} \cdot x_1^{\beta_1} \dots x_n^{\beta_n} \right] \oplus a_0 \cdot \left[\bigoplus_{\hat{\alpha} \in \text{Eq}_k^n} x_1^{\alpha_1} \dots x_n^{\alpha_n} \right],$$

where $3 \leq n \leq k$, $\hat{\beta} = (\beta_1, \dots, \beta_n)$ and $\hat{\alpha} = (\alpha_1, \dots, \alpha_n)$, and at least two among the coefficients $a_0, a_{\hat{\beta}} \in K$ are distinct.

Hence if $f \in G_{n,k}^n$ is a symmetric function, then $f(\hat{\alpha}) = f(\hat{\beta})$ for all $\hat{\alpha}, \hat{\beta} \in \text{Eq}_k^n$.

Example 2. 1. Let us consider the function

$$f = x_1^0 \oplus x_2 \oplus x_1^0 x_2^1 x_3^2 x_4^3 \pmod{5}.$$

Since $f_{3 \leftarrow 1} = f_{4 \leftarrow 1} = f_{3 \leftarrow 2} = f_{4 \leftarrow 2} = f_{4 \leftarrow 3} = x_1^0 \oplus x_2 \pmod{5}$ and $f_{2 \leftarrow 1} = x_1^0 \oplus x_1 \pmod{5}$ it follows that $f \in G_{2,5}^4$. Clearly f is not a symmetric function. On the other hand we have $f(0,0,0,0) = 1$, $f(1,1,1,1) = 1$, $f(2,2,2,2) = 2$, $f(3,3,3,3) = 3$ and $f(4,4,4,4) = 4$. Hence $f \in G_{2,5}^4$, but f does not satisfy Lemma 4.

2. Let us consider the function

$$f = x_1 \oplus x_2 \oplus x_3 \oplus x_4 \pmod{2}.$$

It is easy to check that $f \in G_{2,2}^4$ and f is a symmetric function. Clearly, $f(0,0,0,0) = 0$ and $f(0,0,1,0) = 1$, i.e. $f(0,0,0,0) \neq f(0,0,1,0)$.

3. Let us consider the function

$$\begin{aligned}
f = & x_1^0 x_2^0 \oplus x_1^0 x_2^1 x_3^0 \oplus x_1^0 x_2^2 x_3^0 \oplus x_1^1 x_2^0 x_3^0 \oplus x_1^2 x_2^0 x_3^0 \oplus 2 \cdot (x_1^1 x_2^1 \oplus x_1^1 x_2^0 x_3^1 \oplus \\
& \oplus x_1^1 x_2^2 x_3^1 \oplus x_1^0 x_2^1 x_3^1 \oplus x_1^2 x_2^1 x_3^1) \pmod{3}.
\end{aligned}$$

It is easy to check that $f \in G_{2,3}^3$ and f is a symmetric function. On the other hand we have $f(0,0,0) = 1$, $f(1,1,1) = 2$ and $f(2,2,2) = 0$. Additionally it is easy to see that $f_{2 \leftarrow 1} = x_1^0 \oplus 2x_1^1$. Hence f does not satisfy Lemma 2 and Lemma 4.

Let $\hat{\alpha} = (\alpha_1, \dots, \alpha_n) \in K^n$ and $\hat{\beta} = (\beta_1, \dots, \beta_m) \in K^m$ with $m \leq n$.

We shall write $\hat{\beta} \leq \hat{\alpha}$ if there are $1 \leq i_1, \dots, i_m \leq n$ such that $\alpha_{i_j} = \beta_j$ and $\alpha_s \neq \beta_j$ for all $s \notin \{i_1, \dots, i_m\}$ and $j \in \{1, \dots, m\}$.

Let us denote

$$S(\hat{\alpha}) = \bigoplus_{\pi \in S_n} x_1^{\alpha_{\pi(1)}} \dots x_n^{\alpha_{\pi(n)}}.$$

Example 3. Let $k = 5$. Then $(0, 1, 1) \leq (0, 1, 2, 1, 4)$, but $(0, 1) \not\leq (0, 1, 2, 1, 4)$ and $(0, 2, 3) \not\leq (0, 1, 2, 1, 4)$. Let $\hat{\alpha} = (1, 2, 4)$. Then

$$S(\hat{\alpha}) = x_1^1 x_2^2 x_3^4 \oplus x_1^1 x_2^4 x_3^2 \oplus x_1^2 x_2^1 x_3^4 \oplus x_1^2 x_2^4 x_3^1 \oplus x_1^4 x_2^1 x_3^2 \oplus x_1^4 x_2^2 x_3^1.$$

Lemma 5. *Let $f \in G_{2,k}^3$. Then f is a symmetric function if and only if it can be represented in one of the following forms:*

$$(5) \quad f = \bigoplus_{i=0}^{k-1} a_i [x_1^i x_2^i x_3^i \oplus \bigoplus_{\hat{\alpha} \in Eq_k^3, (i) \leq \hat{\alpha}} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}] \oplus \bigoplus_{\hat{\delta} \in Dis_k^3} b_{\hat{\delta}} S(\hat{\delta})$$

or

$$(6) \quad f = \bigoplus_{i=0}^{k-1} a_i [x_1^i x_2^i x_3^i \oplus \bigoplus_{\hat{\alpha} \in Eq_k^3, (ii) \leq \hat{\alpha}} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}] \oplus \bigoplus_{\hat{\delta} \in Dis_k^3} b_{\hat{\delta}} S(\hat{\delta}),$$

where $\hat{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $b_{\hat{\delta}} \in K$ and at least two among the coefficients $a_i \in K$, for $i = 0, \dots, k-1$ are distinct.

Proof. If $k = 2$ then the lemma directly follows from (2).

Without any difficult, excluding the more complex calculations, are might generalize the results from $G_{2,3}^3$ in Theorem 5.1 [7] to $G_{2,k}^3$ for arbitrary k , $k \geq 3$. In this case, we obtain the same conjunctions of three disjunctions to determine the functions from $G_{2,k}^3$. The difference is that in the equations (13) – (15) (proof of Theorem 5.1 [7]) participate k tuples of k coefficients. Then a function belongs to $G_{2,k}^3$ if and only if it can be represented as in (9) – (12) (Theorem 5.1 [7]) as the sums are extended up to $k-1$ instead of 2 in the case $G_{2,3}^3$. The following two equations generalize in this way (10) and (11) from the proof of Theorem 5.1 [7].

$$(7) \quad f = \bigoplus_{i=0}^{k-1} a_i [x_1^i x_2^i \oplus x_1^i u^{(i)}(x_2, x_3) \oplus x_2^i u^{(i)}(x_1, x_3)] \oplus p_3(x_1, x_2, x_3),$$

$$(8) \quad f = \bigoplus_{i=0}^{k-1} a_i [x_1^i x_2^i \oplus x_2^i v^{(i)}(x_3, x_1) \oplus x_2^i u^{(i)}(x_1, x_3)] \oplus p_3(x_1, x_2, x_3),$$

where $p_3 \in G_{3,k}^3$, at least two among the coefficients a_i , for $i = 0, \dots, k-1$ are distinct, $g \in G_{3,k}^3$ is symmetric and

$$s(x_1, x_2) := \bigoplus_{i=0}^{k-1} x_1^i x_2^i, \quad u^{(i)}(x_1, x_2) := \bigoplus_{j \neq i} x_1^j x_2^j, \quad v^{(i)}(x_1, x_2) := \bigoplus_{j \neq i} x_1^i x_2^j.$$

It is easy to see that if f is represented as in (7) then $f(0,0,1) = a_0$ and $f(1,0,0) = f(0,1,0) = a_1$. If f is given by (8) then $f(0,1,0) = a_1$ and $f(1,0,0) = f(0,0,1) = a_0$. Hence f is not symmetric in the both representations (7) and (8).

Thus the symmetric functions in $G_{2,k}^3$ might be obtained by the equations (9) and (12) in [7] with additional requirement - the functions p_3 have to be symmetric. The symmetry of f then can be obtained as an easy calculation. Hence $f \in G_{2,k}^3$ is a symmetric function if and only if

$$(9) \quad f = \bigoplus_{i=0}^{k-1} a_i [x_3^i s(x_1, x_2) \oplus x_2^i u^{(i)}(x_1, x_3) \oplus x_1^i u^{(i)}(x_2, x_3)] \oplus g(x_1, x_2, x_3)$$

or

$$(10) \quad f = \bigoplus_{i=0}^{k-1} a_i [x_1^i x_2^i \oplus x_1^i v^{(i)}(x_3, x_2) \oplus x_2^i v^{(i)}(x_3, x_1)] \oplus g(x_1, x_2, x_3),$$

where at least two among the coefficients a_i , for $i = 0, \dots, k-1$ are distinct and $g \in G_{3,k}^3$ is a symmetric function. Consequently, g has to be represented as in (12). So, without any difficult from (9), (10) and (12) we might obtain (5) and (6). \square

Theorem 1. *If $f \in G_{2,k}^n$ is a symmetric function with $2 \leq n$ and n is an odd natural number, then there are two constants $c_1, c_2 \in K$ such that*

$$f(c_1, \dots, c_1) \neq f(c_2, \dots, c_2).$$

Proof. Let $n = 3$. Then from (5) and (6) it follows that $f(i, i, i) = a_i$ and at least two among the coefficients a_i , $i = 0, \dots, k-1$ are distinct. Hence there are two constants $c_1, c_2 \in K$ such that $f(c_1, c_1, c_1) \neq f(c_2, c_2, c_2)$, because at least two among the coefficients a_i , for $i = 0, \dots, k-1$ are distinct (see Lemma 5).

Let $n > 3$ is an odd natural number. Let us suppose that this is not the case, i.e.

$$(11) \quad f(c_1, c_1, c_1, \dots, c_1) = f(c_2, c_2, c_2, \dots, c_2)$$

for all $c_1, c_2 \in K$. From Lemma 3 (if $n \leq k$) and Theorem 2.1 [8] (if $n > k$) it follows that all identification minors of f are symmetric functions, which depend essentially on $n-2$ variables. Hence there is a symmetric function $h_1 \in G_{2,k}^{n-2}$ such that $h_1 = f(x_1, \dots, x_{n-2}, c, c)$, for all $c \in K$ because $x_{n-1} \notin \text{Ess}(f_{n \leftarrow n-1})$, according to Lemma 2. If $n-2 > 3$, then again, as above, it follows that there exists a symmetric function $h_2 \in G_{2,k}^{n-4}$ such that $h_2 = h_1(x_1, \dots, x_{n-4}, c, c)$, for all $c \in K$ because $x_{n-3} \notin \text{Ess}([h_1]_{n-2 \leftarrow n-3})$.

In the same way it follows that there is a symmetric function

$$g = h_{\frac{n-3}{2}}(x_1, x_2, x_3) = h_{\frac{n-5}{2}}(x_1, x_2, x_3, c, c) \in G_{2,k}^3$$

for all $c \in K$ because $x_4 \notin \text{Ess}([h_{\frac{n-5}{2}}]_{5 \leftarrow 4})$. Now, (11) implies that $g(c_1, c_1, c_1) = g(c_2, c_2, c_2)$ for all $c_1, c_2 \in K$ which contradicts the case $p = 2$ and $n = 3$. \square

Corollary 1. *If $f \in G_{2,k}^n$, $3 \leq n$ is a symmetric function and n is an odd natural number, then for each $\hat{\alpha} \in K^n$ there is $c_{\hat{\alpha}} \in K$ such that $f(\hat{\alpha}) \neq f(c_{\hat{\alpha}}, c_{\hat{\alpha}}, \dots, c_{\hat{\alpha}})$.*

As usual we shall say that a k -valued function $f \in P_k^n$ is *linear* if $f = a_1x_1 \oplus a_2x_2 \oplus \dots \oplus a_nx_n \oplus c$, where $a_1, a_2, \dots, a_n, c \in K$. Clearly, $x_i \in \text{Ess}(f)$ if and only if $a_i \neq 0$.

Theorem 2. *The set P_k^n contains a linear function with non-trivial arity gap if and only if $k, k \geq 2$, is an even natural number.*

Proof. Obviously, if f is a linear function then $\text{gap}(f) \leq 2$.

Let k be an even natural number and $k = 2m$ for some $m \in \mathbb{N}$. Then let us consider the following linear (and symmetric) function

$$f = m(x_1 \oplus x_2 \oplus \dots \oplus x_n) \oplus c,$$

for some $c \in K$. Clearly,

$$f_{i \leftarrow j} = m(x_1 \oplus \dots \oplus x_{j-1} \oplus x_{j+1} \oplus \dots \oplus x_{i-1} \oplus x_{i+1} \oplus \dots \oplus x_n) \oplus c.$$

Hence $f \in G_{2,k}^n$.

Let k be an odd natural number and $f = a_1x_1 \oplus \dots \oplus a_nx_n \oplus c$, for some $c \in K$, be a linear k -valued function. First assume that there are i and j , $1 \leq i, j \leq n$ such

that $i \neq j$ and $a_i = a_j \neq 0$. Without loss of generality let us assume $(j, i) = (1, 2)$. Then we have

$$f_{2 \leftarrow 1} = 2a_1x_1 \oplus a_3x_3 \oplus \dots \oplus a_nx_n \oplus c.$$

Since k is odd it follows that $2a_1 \neq 0 \pmod{k}$, i.e. $Ess(f_{2 \leftarrow 1}) = \{x_1, \dots, x_n\} \setminus \{x_2\}$. Hence $f \notin G_{2,k}^n$. Second, let $a_i \neq a_j$ for all i and j , $1 \leq i, j \leq n$. Then we have $a_1 + a_2 \neq k$ or $a_1 + a_3 \neq k$. Without loss of generality assume that $a_1 + a_2 \neq k$. Hence

$$f_{2 \leftarrow 1} = (a_1 + a_2)x_1 \oplus a_3x_3 \oplus \dots \oplus x_n \oplus c.$$

Since $k \neq a_1 + a_2 < 2k$ it follows that $a_1 + a_2 \neq 0 \pmod{k}$ which implies $f \notin G_{2,k}^n$. \square

Example 4. Let

$$f = 2(x_1 \oplus x_2 \oplus x_3 \oplus x_4) \pmod{5}.$$

Then the arity gap of f is trivial, according to Theorem 2. In the same time the essential arity gap of the function

$$g = 2(x_1 \oplus x_2 \oplus x_3 \oplus x_4) \pmod{4}$$

is equal to 2. It is easy to check that

$$g(x_1 = 1) = g(x_1 = 3) = 2(x_2 \oplus x_3 \oplus x_4) \oplus 2 \pmod{4}$$

and

$$g(x_1 = 0) = g(x_1 = 2) = 2(x_2 \oplus x_3 \oplus x_4) \pmod{4}.$$

Theorem 3. Let $f \in G_{n,k}^n$, $2 < n \leq k$. Then f is a symmetric function if and only if it can be represented in the following form:

$$(12) \quad f = \left[\bigoplus_{i=1}^m a_i S(\hat{\beta}_i) \right] \oplus a_0 \left[\bigoplus_{\hat{\alpha} \in Eq_k^n} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \right],$$

where $m = \binom{k}{n}$ and $\hat{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\hat{\beta}_i \in Dis_k^n$, and at least two among the coefficients $a_i \in K$, for $i = 1, \dots, k-1$ are distinct.

Proof. Let $f \in G_{n,k}^n$, $2 < n \leq k$ be a symmetric function and let $\hat{\alpha} = (\alpha_1, \dots, \alpha_n) \in Dis_k^n$ be an arbitrary n -tuple of constants from K with $\alpha_i \neq \alpha_j$ when $i \neq j$. Denote by $a_{\hat{\alpha}} = f(\hat{\alpha})$. By the symmetry of f it follows that

$$f(\alpha_{\pi(1)}, \alpha_{\pi(2)}, \dots, \alpha_{\pi(n)}) = a_{\hat{\alpha}},$$

for each $\pi \in S_n$. Since there are $\binom{k}{n}$ ways to choice $\hat{\alpha}$, then the number of distinct coefficients $a_{\hat{\alpha}}$ is at most $\binom{k}{n}$.

Let $\hat{\beta} \in Eq_k^n$. Then (4) implies $f(\hat{\beta}) = f(0, 0, \dots, 0) = a_0$, which proves that f is represented in the form (12).

Clearly, if f is represented as in (12), then it is symmetric and all its identification minors are equal to the constant $f(0, 0, \dots, 0) = a_0$. \square

Corollary 2. There are $k \binom{k}{n}^{+1} - k$ different symmetric functions in $G_{n,k}^n$.

Proof. There exists $\binom{k}{n}$ ways to choose $\hat{\beta}_i$ in (12). Thus there are $\binom{k}{n} + 1$ coefficients in (12), including a_0 . On the other hand we have to exclude all k cases when a_i are the same for $i = 0, \dots, \binom{k}{n}$. \square

3. SUBFUNCTIONS OF SYMMETRIC FUNCTIONS WITH NON-TRIVIAL ARITY GAP

In this section we shall study the subfunctions of the symmetric k -valued functions f with non-trivial arity gap, i.e. with $\text{gap}(f) > 1$.

Definition 6. Let $c \in K$ be a constant from K and $f \in P_k^n$ be a symmetric function. We say that c is the dominant of f if $f(\alpha_1, \dots, \alpha_{n-1}, c) = f(\beta_1, \dots, \beta_{n-1}, c)$ for every constants $\alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_{n-1} \in K$.

Clearly if c is a dominant of f then $\text{Ess}(f(x_1, \dots, x_{n-1}, c)) = \emptyset$, i.e. the subfunctions of f of order 1 obtained by dominants of f are always constant functions. If $f \in G_{n,k}^n$ then c is a dominant of f if and only if $f(\alpha_1, \dots, \alpha_{n-1}, c) = f(0, \dots, 0)$ for all $\alpha_1, \dots, \alpha_{n-1} \in K$, according to (4).

Theorem 4. Let $f \in G_{n,k}^n$ be a symmetric function with $2 \leq k$, $2 < n$ and let $g = f(x_i = c)$ for some x_i , $1 \leq i \leq n$ and for some constant $c \in K$ be a subfunction of f . If c is not a dominant of f then g is a symmetric function which belongs to the class $G_{n-1,k}^{n-1}$.

Proof. We shall consider the non-trivial case $n > 2$ (else the subfunctions of f will depend on at most one essential variable). Hence $k > 2$ because $n = \text{gap}(f) \leq k$.

By symmetry we may assume that $g = f(x_n = c)$.

First we shall prove that $\text{ess}(g) = n - 1$.

According to (4) and by the symmetry of f it follows that if $x_1 \in \text{Ess}(g)$ then $\{x_1, \dots, x_{n-1}\} \subset \text{Ess}(g)$. Thus we shall prove that $x_1 \in \text{Ess}(g)$. Since c is not a dominant of f , from Corollary 1 of Theorem 1, we might assume that there is a $(n-1)$ -tuple $\hat{\delta} = (\delta_1, \dots, \delta_{n-1}) \in \text{Dis}_k^{n-1}$ such that

$$f(\delta_1, \dots, \delta_{n-1}, c) \neq f(0, 0, \dots, 0).$$

Let $f(\delta_1, \dots, \delta_{n-1}, c) = a_{\hat{\delta}}$, i.e. $g(\delta_1, \dots, \delta_{n-1}) = a_{\hat{\delta}} \neq a_0$. On the other hand we have $(c, \delta_1, \dots, \delta_{n-1}, c) \in \text{Eq}_k^n$ and according to (4) we obtain

$$f(c, \delta_2, \dots, \delta_{n-1}, c) = f(0, 0, \dots, 0) = a_0.$$

Consequently $g(c, \delta_2, \dots, \delta_{n-1}) = a_0$ and

$$g(c, \delta_2, \dots, \delta_{n-1}) \neq g(\delta_1, \delta_2, \dots, \delta_{n-1}),$$

which proves that $x_1 \in \text{Ess}(g)$.

Second, we shall prove that g is a symmetric function, all its identification minors are equal to the constant $f(0, \dots, 0)$ and $g \in G_{n-1,k}^{n-1}$.

We know that $\text{Ess}(g) = \{x_1, \dots, x_{n-1}\}$. Since f is symmetric we have

$$f(x_1, x_2, x_3, \dots, x_{n-1}, c) = f(x_2, x_1, x_3, \dots, x_{n-1}, c)$$

which proves that $g = f(x_1, x_2, x_3, \dots, x_{n-1}, c)$ is a symmetric function.

Hence $f_{i \leftarrow j} = a_0$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$, where $a_0 = f(0, \dots, 0)$, according to (4). Consequently, $g_{i \leftarrow j} = a_0$ for all $i, j \in \{1, \dots, n-1\}$ with $i \neq j$ which shows that all identification minors of g are equal to $f(0, \dots, 0)$. Hence $g \in G_{n-1,k}^{n-1}$. \square

Let us denote

$$sub_k^n = \binom{n}{1} \cdot \binom{k}{1} + \binom{n}{2} \cdot \binom{k}{2} + \dots + \binom{n}{n-2} \cdot \binom{k}{n-2}.$$

Lemma 6. *If $f \in G_{n,k}^n$, $n \leq k$ is a symmetric function, then there are at most sub_k^n subfunctions of f with non trivial essential arity gap.*

Proof. The equation (4) implies that if $g = f(c_1, \dots, c_m, x_{m+1}, \dots, x_n)$ is a subfunction of f of order m , $m > 1$ with $(c_1, \dots, c_m) \in Eq_k^m$ then $g = f(0, \dots, 0)$, i.e. g is a constant and hence its arity gap can not be non-trivial. Thus we are interesting in subfunctions g of f obtained when $(c_1, \dots, c_m) \in Dis_k^m$ for $m = 2, \dots, n-2 < k$.

It is easy to see that $|Dis_k^m| = \binom{k}{m}$ for $m = 2, \dots, n-2 < k$.

Consequently, each n -ary k -valued symmetric function has at most $\binom{n}{r} \cdot \binom{k}{r}$ subfunctions of order r , $n-2 \geq r \geq 1$ with non-trivial essential arity gap. Note that symmetry of f implies

$$f(c_1, \dots, c_m, x_{m+1}, \dots, x_n) = f(c_{\pi(1)}, \dots, c_{\pi(m)}, x_{m+1}, \dots, x_n)$$

for each permutation $\pi \in S_m$.

Thus, if $f \in G_{n,k}^n$, $n \leq k$ is a symmetric function then the number of all its non-trivial(non-constant) subfunctions is at most sub_k^n . \square

Theorem 5. *If $f \in G_{2,k}^n$ is a symmetric function with $2 \leq k$ and $2 < n$, then each its subfunction g of order 1 is symmetric and belongs to the class $G_{2,k}^{n-1}$.*

Proof. Without loss of generality we may assume that $g := f(x_1 = c)$ for a constant $c \in K$.

Claim 1. $Ess(g) = \{x_2, \dots, x_n\}$.

We shall prove the claim by considering cases:

Case A. Let $k = 2$ and $n = 3$. Then from (3) it follows that g depends essentially on x_2 and x_3 .

Case B. Let $k = 2$ and $n > 3$. Then from Theorem 3.4 [5] it follows that a Boolean function $f \in P_2^n$, depending on n essential variables with $n \geq 4$, has essential arity gap 2 if and only if

$$(13) \quad f = \bigoplus_{\Sigma \alpha_i \text{ is odd}} x_1^{\alpha_1} \dots x_n^{\alpha_n} \quad \text{or} \quad f = \bigoplus_{\Sigma \alpha_i \text{ is even}} x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

Then we have

$$(14) \quad g = \bigoplus_{\Sigma \alpha_i \text{ is even}} x_2^{\alpha_2} \dots x_n^{\alpha_n} \quad \text{or} \quad g = \bigoplus_{\Sigma \alpha_i \text{ is odd}} x_2^{\alpha_2} \dots x_n^{\alpha_n}.$$

It is clear that the subfunction $g = f(x_1 = c)$ in the both its presentations depends essentially on all of its variables $\{x_2, \dots, x_n\}$.

Case C. Let $k > 2$ and $n = 3$. From (5) and (6) we obtain

$$(15) \quad f(x_1 = c) = \bigoplus_{i=0}^{k-1} a_c[x_2^i x_3^i] \oplus \left[\bigoplus_{j=0, j \neq c}^{k-1} a_j[x_2^j x_3^c \oplus x_2^c x_3^j] \right] \oplus \left[\bigoplus_{\hat{\delta} \in Dis_k^2} b_{\hat{\delta}} S(\hat{\delta}) \right]$$

or

$$(16) \quad f(x_1 = c) = \bigoplus_{i=0}^{k-1} a_i [x_2^i x_3^i] \oplus a_c \left[\bigoplus_{\hat{\alpha} \in Dis_k^2, (i) \leq \hat{\alpha}} x_2^{\alpha_2} x_3^{\alpha_3} \right] \oplus \left[\bigoplus_{\hat{\delta} \in Dis_k^2} b_{\hat{\delta}} S(\hat{\delta}) \right],$$

where $\hat{\alpha} = (\alpha_2, \alpha_3)$, $b_{\hat{\delta}} \in K$ and at least two among the coefficients $a_i \in K$, for $i = 0, \dots, k-1$ are distinct. So it is clear that in the both representations of f the function $g = f(x_1 = c)$ depends essentially on x_2 and x_3 .

Case D. Let $k > 2$ and $n > 3$. Let h be a subfunction of g of order 1, such that $h = g(x_n = c)$, i.e. $h = f(c, x_2, \dots, x_{n-1}, c)$. Hence $h = f_{n \leftarrow 1}$ is an identification minor of f . From Theorem 2.1 [8] (if $n > k$) and Lemma 3 (if $n \leq k$) it follows that h depends essentially on all of its variables $\{x_2, \dots, x_{n-1}\}$. Clearly $Ess(h) \subseteq Ess(g)$ and $\{x_2, \dots, x_{n-1}\} \subseteq Ess(g)$. If we proceed with x_{n-1} as with x_n above, then we shall obtain $\{x_2, \dots, x_{n-2}, x_n\} \subseteq Ess(g)$.

Hence in all these cases we have $\{x_2, \dots, x_{n-1}, x_n\} = Ess(g)$ and $ess(g) = n-1$.

Claim 2. $x_v \notin Ess(g_{u \leftarrow v})$ for all distinct $x_u, x_v \in Ess(g)$.

First, let $k = 2$. If $n = 3$ then from (3) it follows that $x_2 \notin Ess(g_{3 \leftarrow 2})$. If $n > 3$ then from (13) it is easy to check that $x_v \notin Ess(g_{u \leftarrow v})$ for all $1 \leq u, v \leq n$, $u \neq v$.

Second, let $k > 2$. If $n > k$ then $x_v \notin Ess(f(x_1 = c)_{u \leftarrow v})$ for all distinct $x_u, x_v \in \{x_2, \dots, x_n\}$ according to Theorem 2.1 [8]. Finally, let $k > 2$ and $n \leq k$. Then the claim follows from Lemma 2.

Claim 3. g is symmetric and all its identification minors are symmetric functions.

Claim 1 implies that $Ess(g) = \{x_2, \dots, x_n\}$. Since f is symmetric we obtain

$$f(c, x_2, x_3, x_4, \dots, x_n) = f(c, x_3, x_2, x_4, \dots, x_n)$$

which proves that $g = f(c, x_2, x_3, x_4, \dots, x_n)$ is a symmetric function.

From (3) and (14) (if $k = 2$), Theorem 2.1 [8] (if $n > k$), (15) and (16) (if $n = 3$ and $k > 2$) and Lemma 3 (if $3 < n \leq k$) it follows that all identification minors of f are symmetric. Consequently, $f_{i \leftarrow j}$ is symmetric for all $i, j \in \{2, \dots, n\}$ with $i \neq j$. Hence $g_{i \leftarrow j} = f_{i \leftarrow j}(x_1 = c)$ are also symmetric functions for all $i, j \in \{2, \dots, n\}$ with $i \neq j$.

Claim 4. $g \in G_{2,k}^{n-1}$.

Claim 2 implies $gap(g) \geq 2$. According to Claim 3 g is a symmetric function. From Lemma 1 it follows that $gap(g) = 2$ or $gap(g) = n-1$.

If $k = 2$ and $n = 3$, then from (3) it follows that $g \in G_{2,2}^2$.

If $k = 2$ and $n > 3$, then from (14) it follows that $g \in G_{2,2}^n$.

If $3 = n \leq k$ then (15) and (16) imply $g \in G_{2,k}^3$.

If $k > 2$, then Lemma 3 and Theorem 2.1 [8] provide that the subfunctions of f of order 1 are symmetric and their essential arity gap is equal to 2. \square

Corollary 3. If $f \in G_{2,k}^n$ is a symmetric function, then there do not exist any dominants of f .

Example 5. Let us consider the following function

$$f = S(0, 1, 2, 3) \oplus S(0, 1, 4, 5) \oplus 8.S(0, 1, 6, 7)(mod\ 9).$$

Clearly, $f \in G_{4,9}^4$. The constant 8 is a dominant of f because

$$f(\alpha_1, \alpha_2, \alpha_3, 8) = f(0, 0, 0, 0) = 0, \text{ for } 0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 8.$$

The subfunctions of f might be explicitly determined. Since

$$\begin{aligned}
 f(x_4 = 0) &= S(1, 2, 3) \oplus S(1, 4, 5) \oplus 8.S(1, 6, 7), \\
 f(x_4 = 1) &= S(0, 2, 3) \oplus S(0, 4, 5) \oplus 8.S(0, 6, 7), \\
 f(x_4 = 2) &= S(0, 1, 3), \quad f(x_4 = 3) = S(0, 1, 2), \\
 f(x_4 = 4) &= S(0, 1, 5), \quad f(x_4 = 5) = S(0, 1, 4), \\
 f(x_4 = 6) &= 8.S(0, 1, 7), \quad f(x_4 = 7) = 8.S(0, 1, 6),
 \end{aligned}$$

there are $8.4 = 32$ subfunctions of f of order 1 with non-trivial arity gap. All they are symmetric. Since

$$\begin{aligned}
 f(x_4 = 0, x_3 = 2) &= S(1, 3), \quad f(x_4 = 0, x_3 = 3) = S(1, 2), \\
 f(x_4 = 0, x_3 = 4) &= S(1, 5), \quad f(x_4 = 0, x_3 = 5) = S(1, 4), \\
 f(x_4 = 1, x_3 = 2) &= S(0, 3), \quad f(x_4 = 1, x_3 = 3) = S(0, 2), \\
 f(x_4 = 1, x_3 = 4) &= S(0, 5), \quad f(x_4 = 1, x_3 = 5) = S(0, 4), \\
 f(x_4 = 0, x_3 = 6) &= 8.S(1, 7), \quad f(x_4 = 0, x_3 = 7) = 8.S(1, 6), \\
 f(x_4 = 1, x_3 = 6) &= 8.S(0, 7), \quad f(x_4 = 1, x_3 = 7) = 8.S(0, 6), \\
 f(x_4 = 6, x_3 = 7) &= 8.S(0, 1), \\
 f(x_4 = 0, x_3 = 1) &= S(2, 3) \oplus S(4, 5) \oplus 8.S(6, 7), \\
 f(x_4 = 2, x_3 = 3) &= f(x_4 = 4, x_3 = 5) = S(0, 1),
 \end{aligned}$$

there are $16.6 = 96$ subfunctions of f of order 2 with non-trivial arity gap. All they are symmetric. Hence the number of the all subfunctions of f with non trivial arity gap is $32 + 96 = 128$.

There are 4 constant-subfunctions of f of order 1, obtained as follows $f(x_i = 8) = 0$ for $i = 1, 2, 3, 4$.

Since $f(x_i = 8, x_j = m) = 0$ for all $m = 1, \dots, 8$, $i, j = 1, 2, 3, 4$ with $i \neq j$ and

$$\begin{aligned}
 f(x_4 = 2, x_3 = 4) &= f(x_4 = 2, x_3 = 5) = f(x_4 = 2, x_3 = 6) = \\
 f(x_4 = 2, x_3 = 7) &= f(x_4 = 3, x_3 = 4) = f(x_4 = 3, x_3 = 5) = \\
 f(x_4 = 3, x_3 = 6) &= f(x_4 = 3, x_3 = 7) = f(x_4 = 4, x_3 = 6) = \\
 f(x_4 = 4, x_3 = 7) &= f(x_4 = 5, x_3 = 6) = f(x_4 = 5, x_3 = 7) = 0,
 \end{aligned}$$

there are $20.6 = 120$ constant-subfunctions of f of order 2. Thus we have

$$128 + 4 + 120 = \binom{4}{1} \cdot \binom{9}{1} + \binom{4}{2} \cdot \binom{9}{2} = 252 = sub_9^4.$$

4. SEPARABLE SETS OF SYMMETRIC FUNCTIONS WITH NON-TRIVIAL ARITY GAP

Clearly, \prec is a partial order relation in the set P_k^n .

Theorem 6. *If f is a symmetric function with non-trivial arity gap, then each set of essential variables in f is separable in f .*

Proof. Let $f \in G_{n,k}^n$, $n \leq k$ and let $Ess(f) := \{x_1, \dots, x_n\}$ and without loss of generality let us prove that $M = \{x_1, \dots, x_m\}$, $m < n$ is a separable set in f . According to (4) there are constants $c_1, \dots, c_n \in K$ such that $f(c_1, \dots, c_n) \neq a_0$, where $a_0 = f(d_1, \dots, d_n)$ for all $d_1, \dots, d_n \in Eq_k^n$. We have to prove that $M = Ess(f_1)$ where $f_1 = f(x_{m+1} = c_{m+1}, \dots, x_n = c_n)$. Let $x_t \in M$ be an arbitrary variable from M , i.e. $1 \leq t \leq m$. Again from (4) it follows that

$$f(c_1, \dots, c_{t-1}, c_n, c_{t+1}, \dots, c_m, \dots, c_n) = a_0.$$

Hence $x_t \in Ess(f_1)$ which implies $M = Ess(f_1)$.

Let $f \in G_{2,k}^n$, $n \leq k$ be a symmetric function. Without loss of generality let us assume that $M = \{x_1, \dots, x_m\}$, $m < n$ be a set of essential variables in f . We have to prove that M is a separable set in f . Since $x_1 \in Ess(f)$ by Theorem 1.2 [2] it

follows that there is a chain of subfunctions

$$f_1 \prec f_2 \prec \dots \prec f_n = f$$

such that $Ess(f_1) = \{x_1\}$ and $Ess(f_j) = \{x_1, x_{i_2}, \dots, x_{i_j}\}$ for $j = 2, 3, \dots, n$. Since f is a symmetric function, then there are constants c_{m+1}, \dots, c_n for the variables in $Ess(f) \setminus Ess(f_m)$ such that

$$\begin{aligned} f_m &= f(x_{i_{m+1}} = c_{m+1}, x_{i_{m+2}} = c_{m+2}, \dots, x_{i_n} = c_n) = \\ &= f(x_{m+1} = c_{m+1}, x_{m+2} = c_{m+2}, \dots, x_n = c_n). \end{aligned}$$

Consequently, $f(x_{m+1} = c_{m+1}, x_{m+2} = c_{m+2}, \dots, x_n = c_n)$ is a function which depends essentially on the variables x_1, \dots, x_m , i.e. M is a separable set in f . \square

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